Relative Singularity Categories [*†]

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Abstract

We study the properties of the relative derived category $D^b_{\mathscr{C}}(\mathscr{A})$ of an abelian category $\mathscr{A}$ relative to a full and additive subcategory $\mathscr{C}$. In particular, when $\mathscr{A} = A$- mod for a finite-dimensional algebra A over a field and $\mathscr{C}$ is a contravariantly finite subcategory of A-mod which is admissible and closed under direct summands, the $\mathscr{C}$-singularity category $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$=$D^b_{\mathscr{C}}(\mathscr{A})/K^b(\mathscr{C})$ is studied. We give a sufficient condition when this category is triangulated equivalent to the stable category of the Gorenstein category $\mathscr{G}(\mathscr{C})$ of $\mathscr{C}$.

1. Introduction

Let A be a finite-dimensional algebra over a field. We denote by A-mod the category of finitely generated left A-modules, and A-proj (resp. A-inj) the full subcategory of A-mod consisting of projective (resp. injective) modules. We use $K^b(A)$ and $D^b(A)$ to denote the bounded homotopy and derived categories of A-mod respectively, and $K^b(A\text{-proj})$ (resp. $K^b(A\text{-inj})$) to denote the bounded homotopy category of A-proj (resp. A-inj).

The composition functor $K^b(A\text{-proj}) \to K^b(A) \to D^b(A)$ with the former functor the inclusion functor and the latter one the quotient functor is naturally a fully faithful triangle functor, and then one can view $K^b(A\text{-proj})$ as a triangulated subcategory of $D^b(A)$. In fact it is a thick one by [Bu, Lemma 1.2.1]. Consider the quotient triangulated category $D_{sg}(A) := D^b(A)/K^b(A\text{-proj})$, which is the so-called "singularity category". This category was first introduced and studied by Buchweitz in [Bu] where A is assumed to be a left and right noetherian ring. Later on Rickard proved in [R] that for a self-injective algebra A, this category is triangle-equivalent to the stable category of A-mod. This result was generalized to Gorenstein algebra by Happel in [H2]. Since A has finite global dimension if and only if $D_{sg}(A) = 0$, from this viewpoint $D_{sg}(A)$ measures the homological singularity of the algebra A, we call it the singularity category after [O].

Besides, other quotient triangulated categories have been studied by many authors. Beligiannis considered the quotient triangulated categories $D^b(R\text{-Mod})/K^b(R\text{-Proj})$ and $D^b(R\text{-Mod})/K^b(R\text{-Inj})$ for arbitrary ring R, where R-Mod is the category of left R-modules and R-Proj (resp. R-Inj) is the full subcategory of R-Mod consisting of projective (resp. injective) modules (see [Be]). Let $\mathscr{A}$ be an abelian category. A full and additive subcategory ω of $\mathscr{A}$ is called *self-orthogonal* if $\mathrm{Ext}^i_{\mathscr{A}}(M, N) = 0$

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for any $M, N \in \omega$ and $i \geq 1$; in particular, an object T in $\mathscr{A}$ is called *self-orthogonal* if $\operatorname{Ext}^i_{\mathscr{A}}(T, T) = 0$ for any $i \geq 1$. Chen and Zhang studied in [CZ] the quotient triangulated category $D^b(A)/K^b(\operatorname{add}_A T)$ for a finite-dimensional algebra A and a self-orthogonal module T in A-mod, where $\operatorname{add}_A T$ is the full subcategory of A-mod consisting of direct summands of finite direct sums of T. Recently Chen studied in [C2] the relative singularity category $D_\omega(\mathscr{A}) := D^b(\mathscr{A})/K^b(\omega)$ for an arbitrary abelian category $\mathscr{A}$ and an arbitrary self-orthogonal, full and additive subcategory ω of $\mathscr{A}$.

For an abelian category $\mathscr{A}$ with enough projective objects, the Gorenstein derived category $D^*_{gp}(\mathscr{A})$ of $\mathscr{A}$ was introduced by Gao and Zhang in [GZ], where $* \in \{\text{blank}, -, b\}$. It can be viewed as a generalization of the usual derived category $D^*(\mathscr{A})$ by using Gorenstein projective objects instead of projective objects and $\mathscr{GP}$-quasi-isomorphisms instead of quasi-isomorphisms, where $\mathscr{GP}$ means "Gorenstein projective". For Gorenstein projective modules and Gorenstein projective objects, we refer to [AuB], [EJ1], [EJ2], [Ho] and [SSW]. Asadollahi, Hafezi and Vahed studied in [AHV] the relative derived category $D^*_{\mathscr{C}}(\mathscr{A})$ for an arbitrary abelian category $\mathscr{A}$ with respect to a contravariantly finite subcategory $\mathscr{C}$, where $* \in \{\text{blank}, -, b\}$, and they pointed out that $K^b(\mathscr{C})$ can be viewed as a triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$.

Given a finite-dimensional algebra A over a field and a full and additive subcategory $\mathscr{C}$ of $\mathscr{A}(= A$-mod) closed under direct summands, it follows from [BD] that $K^b(\mathscr{C})$ is a Krull-Schmidt category and hence can be viewed as a thick triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$. If the quotient triangulated category $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) := D^b_{\mathscr{C}}(\mathscr{A})/K^b(\mathscr{C})$ is considered, then it is natural to ask whether $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ shares some nice properties of $D_{sg}(A)$. The aim of this paper is to study this question.

In Section 2, we give some terminology and some preliminary results.

In Section 3, for an abelian category $\mathscr{A}$ and a full and additive subcategory $\mathscr{C}$ of $\mathscr{A}$, we prove that if $\mathscr{C}$ is admissible, then the composition functor $\mathscr{A} \to K^b(\mathscr{A}) \to D^b_{\mathscr{C}}(\mathscr{A})$ is fully faithful, where the former functor is the inclusion functor and the latter one is the quotient functor. Let $\mathscr{C}$ be a contravariantly finite subcategory of $\mathscr{A}$ and $\mathscr{D} \subseteq \mathscr{A}$ a subclass of $\mathscr{A}$. We introduce a dimension denoted by $\mathscr{C}\mathscr{D}\text{-}\dim M$ which is called the *$\mathscr{C}$-proper $\mathscr{D}$-dimension* of an object M in $\mathscr{A}$. By choosing a left $\mathscr{C}$-resolution $C^\bullet_M$ of M, we get a functor $\operatorname{Ext}^n_{\mathscr{C}}(M, -) := H^n \operatorname{Hom}_{\mathscr{A}}(C^\bullet_M, -)$ for any $n \in \mathbb{Z}$. Then by using the properties of this functor we obtain some equivalent characterizations for $\mathscr{C}\mathscr{C}\text{-}\dim M$ being finite.

In Section 4, we introduce the $\mathscr{C}$-singularity category $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) := D^b_{\mathscr{C}}(\mathscr{A})\ /K^b(\mathscr{C})$, where $\mathscr{A} = A$-mod and $\mathscr{C}$ is a contravariantly finite, full and additive subcategory of $\mathscr{A}$ which is admissible and closed under direct summands. We prove that if $\mathscr{C}\mathscr{C}\text{-}\dim \mathscr{A} < \infty$, then $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$. As a consequence, we get that if A is of finite representation type, then $\mathscr{C}\mathscr{C}\text{-}\dim\mathscr{A} < \infty$ if and only if $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$. Let $\mathscr{G}(\mathscr{C})$ be the Gorenstein category of $\mathscr{C}$ and ε the collection of all $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact complexes of the form: $0 \to L \to M \to N \to 0$ with $L, M, N \in \mathscr{G}(\mathscr{C})$. By [Bü] (or [Q]) $(\mathscr{G}(\mathscr{C}), \varepsilon)$ is an exact category; moreover, it is a Frobenius category with $\mathscr{C}$ the subcategory of projective-injective objects, see [H1]. We prove that if $\mathscr{C}\mathscr{G}(\mathscr{C})\text{-}\dim \mathscr{A} < \infty$, then the natural functor $\theta : \mathscr{G}(\mathscr{C}) \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ induces a triangle-equivalence $\theta' : \underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$, where $\underline{\mathscr{G}(\mathscr{C})}$ is the stable category of $\mathscr{G}(\mathscr{C})$.

2. Preliminaries

Throughout this paper, $\mathscr{A}$ is an abelian category, $C(\mathscr{A})$ is the category of complexes of objects in $\mathscr{A}$, $K^*(\mathscr{A})$ is the homotopy category of $\mathscr{A}$ and $D^*(\mathscr{A})$ is the usual derived category by inverting the quasi-isomorphisms in $K^*(\mathscr{A})$, where $* \in \{\text{blank}, -, b\}$. We will use the formula $\mathrm{Hom}_{K(\mathscr{A})}(X^\bullet, Y^\bullet[n]) = H^n \mathrm{Hom}_{\mathscr{A}}(X^\bullet, Y^\bullet)$ for any $X^\bullet, Y^\bullet \in C(\mathscr{A})$ and $n \in \mathbb{Z}$ (the ring of integers).

Let

$$X^\bullet := \cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \to \cdots$$

be a complex and $f : X^\bullet \to Y^\bullet$ a cochain map in $C(\mathscr{A})$. Recall that $X^\bullet$ is called *acyclic* (or *exact*) if $H^i(X^\bullet) = 0$ for any $i \in \mathbb{Z}$, and f is called a *quasi-isomorphism* if $H^i(f)$ is an isomorphism for any $i \in \mathbb{Z}$.

From now on, we fix a full and additive subcategory $\mathscr{C}$ of $\mathscr{A}$.

Definition 2.1. Let $X^\bullet, Y^\bullet$ and f be as above.

(1) ([EJ2]) $X^\bullet$ in $C(\mathscr{A})$ is called *$\mathscr{C}$-acyclic* or $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-*exact* if the complex $\mathrm{Hom}_{\mathscr{A}}(C, X^\bullet)$ is acyclic for any $C \in \mathscr{C}$. Dually, a $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-*exact complex* is defined.

(2) f is called a *$\mathscr{C}$-quasi-isomorphism* if the cochain map $\mathrm{Hom}_{\mathscr{A}}(C, f)$ is a quasi-isomorphism for any $C \in \mathscr{C}$.

Remark 2.2. (1) We use $\mathrm{Con}(f)$ to denote the mapping cone of $f : X^\bullet \to Y^\bullet$. It is well known that f is a quasi-isomorphism if and only if $\mathrm{Con}(f)$ is acyclic; analogously, f is a $\mathscr{C}$-quasi-isomorphism if and only if $\mathrm{Con}(f)$ is $\mathscr{C}$-acyclic.

(2) We use $\mathscr{P}(\mathscr{A})$ to denote the full subcategory of $\mathscr{A}$ consisting of projective objects. If $\mathscr{A}$ has enough projective objects, then every quasi-isomorphism is a $\mathscr{P}(\mathscr{A})$-quasi-isomorphism; and if $\mathscr{P}(\mathscr{A}) \subseteq \mathscr{C}$, then every $\mathscr{C}$-quasi-isomorphism is a quasi-isomorphism.

We use $K^*_{ac}(\mathscr{A})$ (resp. $K^*_{\mathscr{C}\text{-}ac}(\mathscr{A})$) to denote the full subcategory of $K^*(\mathscr{A})$ consists of acyclic complexes (resp. $\mathscr{C}$-acyclic complexes).

Lemma 2.3. *Let $X^\bullet$ be a complex in $C(\mathscr{A})$. Then $X^\bullet$ is $\mathscr{C}$-acyclic if and only if the complex $\mathrm{Hom}_{\mathscr{A}}(C^\bullet, X^\bullet)$ is acyclic for any $C^\bullet \in K^-(\mathscr{C})$.*

Proof. See [CFH, Lemma 2.4]. □

Lemma 2.4. *(1) Let $C^\bullet$ be a complex in $K^-(\mathscr{C})$ and $f : X^\bullet \to C^\bullet$ a $\mathscr{C}$-quasi-isomorphism in $C(\mathscr{A})$. Then there exists a cochain map $g : C^\bullet \to X^\bullet$ such that fg is homotopic to $\mathrm{id}_{C^\bullet}$.*

(2) Any $\mathscr{C}$-quasi-isomorphism between two complexes in $K^-(\mathscr{C})$ is a homotopy equivalence.

Proof. (1) Consider the distinguished triangle:

$$X^\bullet \xrightarrow{f} C^\bullet \to \mathrm{Con}(f) \to X^\bullet[1]$$

in $K(\mathscr{A})$ with $\mathrm{Con}(f)$ $\mathscr{C}$-acyclic. By applying the functor $\mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, -)$ to it, we get an exact sequence:

$$\mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, X^\bullet) \xrightarrow{\mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, f)} \mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, C^\bullet) \to \mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, \mathrm{Con}(f)).$$

It follows from Lemma 2.3 that $\operatorname{Hom}_{K(\mathscr{A})}(C^\bullet, \operatorname{Con}(f)) \cong H^0 \operatorname{Hom}_{\mathscr{A}}(C^\bullet, \operatorname{Con}(f)) = 0$. So there exists a cochain map $g : C^\bullet \to X^\bullet$ such that fg is homotopic to $\mathrm{id}_{C^\bullet}$.

(2) Let $f : X^\bullet \to Y^\bullet$ be a $\mathscr{C}$-quasi-isomorphism with $X^\bullet, Y^\bullet$ in $K^-(\mathscr{C})$. By (1), there exists a cochain map $g : Y^\bullet \to X^\bullet$, such that fg is homotopic to $\mathrm{id}_{Y^\bullet}$. By (1) again, there exists a cochain map $g' : X^\bullet \to Y^\bullet$, such that gg' is homotopic to $\mathrm{id}_{X^\bullet}$. Thus $f = g'$ in $K(\mathscr{A})$ is a homotopy equivalence. $\square$

Definition 2.5. (1) ([AuR]) Let $\mathscr{C} \subseteq \mathscr{D}$ be subcategories of $\mathscr{A}$. The morphism $f : C \to D$ in $\mathscr{A}$ with $C \in \mathscr{C}$ and $D \in \mathscr{D}$ is called a *right $\mathscr{C}$-approximation* of D if for any morphism $g : C' \to D$ in $\mathscr{A}$ with $C' \in \mathscr{C}$, there exists a morphism $h : C' \to C$ such that the following diagram commutes:

$$\begin{tikzcd} & C' \arrow[dl, dashed, "h"'] \arrow[d, "g"] \\ C \arrow[r, "f"] & D. \end{tikzcd}$$

If each object in $\mathscr{D}$ has a right $\mathscr{C}$-approximation, then $\mathscr{C}$ is called *contravariantly finite* in $\mathscr{D}$.

(2) ([C1]) A contravariantly finite subcategory $\mathscr{C}$ of $\mathscr{A}$ is called *admissible* if any right $\mathscr{C}$-approximation is epic. In this case, every $\mathscr{C}$-acyclic complex is acyclic.

The following definition is cited from [Bü], see also [Q] and [K].

Definition 2.6. Let $\mathscr{B}$ be an additive category. A *kernel-cokernel pair* (i, p) in $\mathscr{B}$ is a pair of composable morphisms $L \xrightarrow{i} M \xrightarrow{p} N$ such that i is a kernel of p and p is a cokernel of i. If a class ε of kernel-cokernel pairs on $\mathscr{B}$ is fixed, an *admissible monic* (sometimes called *inflation*) is a morphism i for which there exists a morphism p such that $(i, p) \in \varepsilon$. *Admissible epics* (sometimes called *deflations*) are defined dually.

An exact category is a pair $(\mathscr{B}, \varepsilon)$ consisting of an additive category $\mathscr{B}$ and a class of kernel-cokernel pairs ε on $\mathscr{B}$ with ε closed under isomorphisms satisfying the following axioms:

[E0] For any object B in $\mathscr{B}$, the identity morphism id_B is both an admissible monic and an admissible epic.

[E1] The class of admissible monics is closed under compositions.

[E1^{op}] The class of admissible epics is closed under compositions.

[E2] The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic.

[E2^{op}] The pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.

Elements of ε are called *short exact sequences* (or *conflations*).

Let $\mathscr{B}$ be a triangulated subcategory of a triangulated category $\mathscr{K}$ and S the compatible multiplicative system determined by $\mathscr{B}$. In the Verdier quotient category $\mathscr{K}/\mathscr{B}$, each morphism $f : X \to Y$ is given by an equivalence class of right fractions f/s or left fractions $s\backslash f$ as presented by $X \overset{s}{\Longleftarrow} Z \xrightarrow{f} Y$ or $X \xrightarrow{f} Z \overset{s}{\Longleftarrow} Y$, where the doubled arrow means $s \in S$.

3. $\mathscr{C}$-derived categories

For a subclass $\mathscr{C}$ of objects in a triangulated category $\mathscr{K}$, it is known that the full subcategory $\mathscr{C}^{\perp} = \{X \in \mathscr{K} \mid \mathrm{Hom}_{\mathscr{K}}(C[n], X) = 0 \text{ for any } C \in \mathscr{C} \text{ and } n \in \mathbb{Z}\}$ is a triangulated subcategory of $\mathscr{K}$ and is closed under direct summands, and hence is thick ([R]). It follows that $K^*_{\mathscr{C}\text{-}ac}(\mathscr{A})$ is a thick subcategory of $K^*(\mathscr{A})$.

Definition 3.1. ([V]) The Verdier quotient category $D^*_{\mathscr{C}}(\mathscr{A}) := K^*(\mathscr{A})/K^*_{\mathscr{C}\text{-}ac}(\mathscr{A})$ is called the *$\mathscr{C}$-derived category* of $\mathscr{A}$, where $* \in \{\text{blank}, -, b\}$.

Example 3.2. (1) If $\mathscr{A}$ has enough projective objects and $\mathscr{C} = \mathscr{P}(\mathscr{A})$, then $D^*_{\mathscr{C}}(\mathscr{A})$ is the usual derived category $D^*(\mathscr{A})$.

(2) If $\mathscr{A}$ has enough projective objects and $\mathscr{C} = \mathscr{G}(\mathscr{A})$ (the full subcategory of $\mathscr{A}$ consisting of Gorenstein projective objects), then $D^*_{\mathscr{C}}(\mathscr{A})$ is the Gorenstein derived category $D^*_{gp}(\mathscr{A})$ defined in [GZ].

(3) Let R be a ring and $\mathscr{A} = R\text{-}\mathrm{Mod}$. If $\mathscr{C} = \mathscr{P}\mathscr{P}(R)$ (the full subcategory of $R\text{-}\mathrm{Mod}$ consisting of pure projective modules), then $D^*_{\mathscr{C}}(\mathscr{A})$ is the pure derived category $D^*_{pur}(\mathscr{A})$ in [ZH].

Proposition 3.3. *([AHV]) (1) $D^-_{\mathscr{C}}(\mathscr{A})$ is a triangulated subcategory of $D_{\mathscr{C}}(\mathscr{A})$, and $D^b_{\mathscr{C}}(\mathscr{A})$ is a triangulated subcategory of $D^-_{\mathscr{C}}(\mathscr{A})$.*

(2) For any $C^\bullet \in K^-(\mathscr{C})$ and $X^\bullet \in C(\mathscr{A})$, there exists an isomorphism of abelian groups:

$$\mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, X^\bullet) \cong \mathrm{Hom}_{D_{\mathscr{C}}(\mathscr{A})}(C^\bullet, X^\bullet).$$

(3) Let $\mathscr{C} \subseteq \mathscr{A}$ be admissible. Then the composition functor $\mathscr{A} \to K^b(\mathscr{A}) \to D^b_{\mathscr{C}}(\mathscr{A})$ is fully faithful, where the former functor is the inclusion functor and the latter one is the quotient functor.

Proof. In the following, each morphism in $D^*_{\mathscr{C}}(\mathscr{A})$ will be denoted by the equivalence class of right fractions, where $* \in \{\text{blank}, -, b\}$.

(1) We only prove the first assertion, the second one can be proved similarly.

Note that $D^-_{\mathscr{C}}(\mathscr{A}) = K^-(\mathscr{A})/K^-(\mathscr{A}) \bigcap K_{\mathscr{C}\text{-}ac}(\mathscr{A})$ and $D_{\mathscr{C}}(\mathscr{A}) = K(\mathscr{A})/K_{\mathscr{C}\text{-}ac}(\mathscr{A})$. By [GM, Proposition 3.2.10], it suffices to show that for any $\mathscr{C}$-quasi-isomorphism $s : Y^\bullet \to X^\bullet$ with $X^\bullet \in K^-(\mathscr{A})$, there exists a morphism $f : Z^\bullet \to Y^\bullet$ with $Z^\bullet \in K^-(\mathscr{A})$ such that sf is a $\mathscr{C}$-quasi-isomorphism.

Suppose $X^n \neq 0$ with $X^i = 0$ for any $i > n$. Then there exists a commutative diagram:

$$\begin{tikzcd}
Z^\bullet : \arrow[d, "f"] & \cdots \arrow[r] & Y^{n-1} \arrow[r] \arrow[d, equal] & Y^n \arrow[r] \arrow[d, equal] & \mathrm{Ker}\, d_Y^{n+1} \arrow[r] \arrow[d] & 0 \\
Y^\bullet : \arrow[d, "s"] & \cdots \arrow[r] & Y^{n-1} \arrow[r] \arrow[d] & Y^n \arrow[r] \arrow[d] & Y^{n+1} \arrow[r] \arrow[d] & \cdots \\
X^\bullet : & \cdots \arrow[r] & X^{n-1} \arrow[r] & X^n \arrow[r] & 0 \arrow[r] & \cdots ,
\end{tikzcd}$$

where $\mathrm{Ker}\, d_Y^{n+1} \to Y^{n+1}$ is the canonical map. Since both f and s are $\mathscr{C}$-quasi-isomorphisms, so is sf.

(2) Consider the canonical map $G : \mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, X^\bullet) \to \mathrm{Hom}_{D_{\mathscr{C}}(\mathscr{A})}(C^\bullet, X^\bullet)$ defined by $G(f) = f/\,\mathrm{id}_{C^\bullet}$. If $G(f) = 0$, then there exists a $\mathscr{C}$-quasi-isomorphism $s : Z^\bullet \to C^\bullet$ such that $fs \sim 0$. By Lemma 2.4(1) there exists a cochain map $g : C^\bullet \to Z^\bullet$ such that $sg \sim \mathrm{id}_{C^\bullet}$, and then $f \sim 0$. On the other hand, let $f/s \in \mathrm{Hom}_{D_{\mathscr{C}}(\mathscr{A})}(C^\bullet, X^\bullet)$, that is, it has a diagram of the form $C^\bullet \overset{s}{\Longleftarrow} Z^\bullet \xrightarrow{f} X^\bullet$, where s is a $\mathscr{C}$-quasi-isomorphism. It follows from Lemma 2.4(1) there exists a cochain map $g : C^\bullet \to Z^\bullet$ such that $sg \sim \mathrm{id}_{C^\bullet}$, which implies that $f/s = (fg)/\,\mathrm{id}_{C^\bullet} = G(fg)$. Thus G is an isomorphism, as desired.

(3) Let $F : \mathscr{A} \to D^b_{\mathscr{C}}(\mathscr{A})$ denote the composition functor, it suffices to show that for any $M, N \in \mathscr{A}$, the map $F : \mathrm{Hom}_{\mathscr{A}}(M, N) \to \mathrm{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(M, N)$ is an isomorphism.

Let $f \in \mathrm{Hom}_{\mathscr{A}}(M, N)$. If $F(f) = 0$, then there exists a $\mathscr{C}$-quasi-isomorphism $s : Z^\bullet \to M$ such that $fs \sim 0$, and then $H^0(f)H^0(s) = 0$. Since $H^0(s)$ is an isomorphism, $f = H^0(f) = 0$. On the other hand, let $f/s \in \mathrm{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(M, N)$, that is, it has a diagram of the form $M \overset{s}{\Longleftarrow} Z^\bullet \xrightarrow{f} N$, where s is a $\mathscr{C}$-quasi-isomorphism. Then $H^0(s) : H^0(Z^\bullet) \to M$ is an isomorphism. Put $g := H^0(f)H^0(s)^{-1} \in \mathrm{Hom}_{\mathscr{A}}(M, N)$. Consider the truncation:

$$U^\bullet := \cdots \to Z^{-2} \xrightarrow{d_Z^{-2}} Z^{-1} \xrightarrow{d_Z^{-1}} \mathrm{Ker}\, d^0 \to 0$$

of $Z^\bullet$ and the canonical map $i : U^\bullet \to Z^\bullet$. Since s is a $\mathscr{C}$-quasi-isomorphism, so is si. We have the following commutative diagram:

$$\begin{tikzcd}
U^\bullet \arrow[r, "i"] \arrow[d] & Z^\bullet \arrow[d, "s"] \\
H^0(Z^\bullet) \arrow[r, "H^0(s)"] & M,
\end{tikzcd}$$

where $U^\bullet \to H^0(Z^\bullet)$ is the canonical map, so $gsi = H^0(f)H^0(s)^{-1}si = fi$. Then we get the following commutative diagram of complexes:

$$\begin{tikzcd}
 & Z^\bullet \arrow[dl, Rightarrow, "s"'] \arrow[dr, "f"] & \\
M & U^\bullet \arrow[l, Rightarrow, "si"] \arrow[u, "i"'] \arrow[r, "fi"] \arrow[d, "si"] & N \\
 & M, \arrow[ul, Rightarrow, "\mathrm{id}_M"] \arrow[ur, "g"'] &
\end{tikzcd}$$

which implies $F(g) = g/\,\mathrm{id}_M = f/s$. $\square$

Set $K^{-,\mathscr{C}b}(\mathscr{C}) := \{X^\bullet \in K^-(\mathscr{C}) \mid$ there exists $n \in \mathbb{Z}$ such that $H^i(\mathrm{Hom}_{\mathscr{A}}(C, X^\bullet)) = 0$ for any $C \in \mathscr{C}$ and $i \leq n\}$.

Proposition 3.4. *([AHV, Theorem 3.3]) If $\mathscr{C}$ is a contravariantly finite subcategory of $\mathscr{A}$, then we have a triangle-equivalence $K^{-,\mathscr{C}b}(\mathscr{C}) \cong D^b_{\mathscr{C}}(\mathscr{A})$.*

In the rest of this section, we always suppose that $\mathscr{C}$ is a contravariantly finite subcategory of $\mathscr{A}$ unless otherwise specified.

Definition 3.5. Let $\mathscr{D}$ be a subclass of objects in $\mathscr{A}$ and $M \in \mathscr{A}$.

(1) A *$\mathscr{C}$-proper $\mathscr{D}$-resolution* of M is a $\mathscr{C}$-quasi-isomorphism $f : D^\bullet \to M$, where $D^\bullet$ is a complex of objects in $\mathscr{D}$ with $D^n = 0$ for any $n > 0$, that is, it has an associated $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact complex $\cdots \to D^{-n} \to D^{-n+1} \to \cdots \to D^0 \xrightarrow{f} M \to 0$.

(2) The *$\mathscr{C}$-proper $\mathscr{D}$-dimension* of M, written $\mathscr{C}\mathscr{D}\text{-}\dim M$, is defined as $\inf\{n \mid$ there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact complex $0 \to D^{-n} \to D^{-n+1} \to \cdots \to D^0 \xrightarrow{f} M \to 0\}$. If no such an integer exists, then set $\mathscr{C}\mathscr{D}\text{-}\dim M = \infty$.

(3) For a class $\mathscr{E}$ of objects of $\mathscr{A}$, the *$\mathscr{C}$-proper $\mathscr{D}$-dimension* of $\mathscr{E}$, written $\mathscr{C}\mathscr{D}\text{-}\dim \mathscr{E}$, is defined as $\sup\{\mathscr{C}\mathscr{D}\text{-}\dim M \mid M \in \mathscr{E}\}$.

Remark 3.6. (1) If $\mathscr{A}$ has enough projective objects and $\mathscr{C} = \mathscr{P}(\mathscr{A})$, then a $\mathscr{C}$-proper $\mathscr{D}$-resolution is just a $\mathscr{D}$-resolution and the $\mathscr{C}$-proper $\mathscr{D}$-dimension of an object $M \in \mathscr{A}$ is just the usual $\mathscr{D}$-dimension $\mathscr{D}\text{-}\dim M$ of M.

(2) If $\mathscr{D} = \mathscr{C}$, then a $\mathscr{C}$-proper $\mathscr{D}$-resolution is just a $\mathscr{C}$-proper resolution. In this case, it is also called a *left $\mathscr{C}$-resolution* and the $\mathscr{C}$-proper $\mathscr{D}$-dimension is the left $\mathscr{C}$-dimension (see [EJ2]).

Let $M \in \mathscr{A}$. Since $\mathscr{C}$ is a contravariantly finite subcategory of $\mathscr{A}$, we may choose a left $\mathscr{C}$-resolution $C^\bullet_M \to M$ of M. Put $\mathrm{Ext}^n_{\mathscr{C}}(M, N) := H^n \mathrm{Hom}_{\mathscr{A}}(C^\bullet_M, N)$ for any $N \in \mathscr{A}$ and $n \in \mathbb{Z}$. Note that $C^\bullet_M$ is isomorphic to M in $D_{\mathscr{C}}(\mathscr{A})$. By Proposition 3.3(1)(2), we have $\mathrm{Ext}^n_{\mathscr{C}}(M, N) = H^n \mathrm{Hom}_{\mathscr{A}}(C^\bullet_M, N) = \mathrm{Hom}_{K(\mathscr{A})}(C^\bullet_M, N[n]) \cong \mathrm{Hom}_{D_{\mathscr{C}}(\mathscr{A})}(C^\bullet_M, N[n]) \cong \mathrm{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(M, N[n])$.

The following is cited from [EJ2, Chapter 8].

Lemma 3.7. (1) For any $M \in \mathscr{A}$, the functor $\mathrm{Ext}^n_{\mathscr{C}}(M, -)$ does not depend on the choices of left $\mathscr{C}$-resolutions of M.

(2) For any $M \in \mathscr{A}$ and $n < 0$, $\mathrm{Ext}^n_{\mathscr{C}}(M, -) = 0$ and there exists a natural equivalence $\mathrm{Hom}_{\mathscr{A}}(M, -) \cong \mathrm{Ext}^0_{\mathscr{C}}(M, -)$ whenever $\mathscr{C}$ is admissible.

(3) If $\mathscr{C}$ is admissible, then every $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact complex $0 \to L \to M \to N \to 0$ induces a long exact sequence $0 \to \mathrm{Hom}_{\mathscr{A}}(N, -) \to \mathrm{Hom}_{\mathscr{A}}(M, -) \to \mathrm{Hom}_{\mathscr{A}}(L, -) \to \cdots \to \mathrm{Ext}^n_{\mathscr{C}}(N, -) \to \mathrm{Ext}^n_{\mathscr{C}}(M, -) \to \mathrm{Ext}^n_{\mathscr{C}}(L, -) \to \mathrm{Ext}^{n+1}_{\mathscr{C}}(N, -) \to \cdots$.

Theorem 3.8. *Let $\mathscr{C}$ be admissible and closed under direct summands, then the following statements are equivalent for any $M \in \mathscr{A}$ and $n \geq 0$.*

(1) $\mathscr{C}\mathscr{C}\text{-}\dim M \leq n$.

(2) $\mathrm{Ext}^i_{\mathscr{C}}(M, N) = 0$ for any $N \in \mathscr{A}$ and $i \geq n + 1$.

(3) $\mathrm{Ext}^{n+1}_{\mathscr{C}}(M, N) = 0$ for any $N \in \mathscr{A}$.

(4) For any left $\mathscr{C}$-resolution $C^\bullet_M \to M$ of M, $\mathrm{Ker}\, d^{-n+1}_{C_M} \in \mathscr{C}$, where $d^{-n+1}_{C_M}$ is the $(-n+1)$st differential of $C^\bullet_M$.

Proof. (1) $\Rightarrow$ (2) Let $0 \to C^{-n} \to C^{-n+1} \to \cdots \to C^0 \to M \to 0$ be a left $\mathscr{C}$-resolution of M. Then $\mathrm{Hom}_{\mathscr{A}}(C^{-i}, N)$ =0 for any $N \in \mathscr{A}$ and $i \geq n + 1$ and the assertion follows.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are trivial.

(3) $\Rightarrow$ (4) Let $\cdots \to C_M^{-n} \xrightarrow{d_{C_M}^{-n}} C_M^{-n+1} \to \cdots \to C_M^0 \to M \to 0$ be a left $\mathscr{C}$-resolution of M. Then we get a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence $0 \to \mathrm{Ker}\, d_{C_M}^{-n} \to C_M^{-n} \to \mathrm{Ker}\, d_{C_M}^{-n+1} \to 0$. Since $\mathrm{Ext}_{\mathscr{C}}^{n+1}(M, \mathrm{Ker}\, d_{C_M}^{-n}) = 0$, $\mathrm{Ext}_{\mathscr{C}}^1(\mathrm{Ker}\, d_{C_M}^{-n+1}, \mathrm{Ker}\, d_{C_M}^{-n}) \cong \mathrm{Ext}_{\mathscr{C}}^{n+1}(M, \mathrm{Ker}\, d_{C_M}^{-n}) = 0$ by the dimension shifting. Applying $\mathrm{Hom}_{\mathscr{A}}(-, \mathrm{Ker}\, d_{C_M}^{-n})$ to the exact sequence $0 \to \mathrm{Ker}\, d_{C_M}^{-n} \to C_M^{-n} \to \mathrm{Ker}\, d_{C_M}^{-n+1} \to 0$, it follows from Lemma 3.7(3) that the sequence splits. So $\mathrm{Ker}\, d_{C_M}^{-n+1}$ is a direct summand of C_M^{-n} and $\mathrm{Ker}\, d_{C_M}^{-n+1} \in \mathscr{C}$. $\square$

4. $\mathscr{C}$-singularity categories

In this section, unless otherwise specified, we always suppose that A is a finite-dimensional algebra over a field, $\mathscr{A} = A$-mod and $\mathscr{C}$ is a full and additive subcategory of $\mathscr{A}$ which is contravariantly finite in $\mathscr{A}$ and is admissible and closed under direct summands.

Recall that an additive category is called a *Krull-Schmidt category* if each of its object X has a decomposition $X \cong X_1 \bigoplus X_2 \bigoplus \cdots \bigoplus X_n$ such that each X_i is indecomposable with a local endomorphism ring. By [BD, Proposition A.2] $K^b(\mathscr{C})$ is a Krull-Schmidt category, so it is closed under direct summands and $K^b(\mathscr{C})$ viewed as a full triangulated subcategory of $D_{\mathscr{C}}^b(\mathscr{A})$ is thick. It is of interest to consider the quotient triangulated category $D_{\mathscr{C}}^b(\mathscr{A})\ /K^b(\mathscr{C})$.

Definition 4.1. We call $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) := D_{\mathscr{C}}^b(\mathscr{A})\ /K^b(\mathscr{C})$ the *$\mathscr{C}$-singularity category.*

Example 4.2. (1) If $\mathscr{C} = A$-proj, then $D_{\mathscr{C}}^b(\mathscr{A})$ is the usual bounded derived category $D^b(\mathscr{A})$ and the $\mathscr{C}$-singularity category $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ is the singularity category $D_{sg}(A)$ which is called the "stabilized derived category" in [Bu].

(2) Let $\mathscr{C} = \mathscr{G}(A)$ (the subcategory of A-mod consisting of Gorenstein projective modules). If $\mathscr{G}(A)$ is contravariantly finite in A-mod, for example, if A is Gorenstein (that is, the left and right self-injective dimensions of A are finite) or $\mathscr{G}(A)$ contains only finitely many non-isomorphic indecomposable modules, then the bounded $\mathscr{C}$-derived category of $\mathscr{A}$, denoted by $D_{\mathscr{G}(A)}^b(\mathscr{A})$, is the *bounded Gorenstein derived category* introduced in [GZ]. The $\mathscr{C}$-singularity category $D_{\mathscr{G}(A)\text{-}sg}(\mathscr{A})$ is the quotient triangulated category $D_{\mathscr{G}(A)}^b(\mathscr{A})\ /K^b(\mathscr{G}(A))$, we call it the *Gorenstein singularity category.*

Given a complex $X^\bullet$ and an integer $i \in \mathbb{Z}$, we denote by $\sigma^{\geq i}X^\bullet$ the complex with X^j in the jth degree whenever $j \geq i$ and 0 elsewhere, and set $\sigma^{>i}X^\bullet := \sigma^{\geq i+1}X^\bullet$. Dually, for the notations $\sigma^{\leq i}X^\bullet$ and $\sigma^{<i}X^\bullet$. Recall that the cardinal of the set $\{X^i \neq 0 \mid i \in \mathbb{Z}\}$ is called the *width* of $X^\bullet$, and denoted by $\omega(X^\bullet)$.

It is well known that A has finite global dimension if and only if $D_{sg}(A) = 0$. For the $\mathscr{C}$-singularity category $D_{\mathscr{C}\text{-}sg}^b(\mathscr{A})$ we have the following property.

Proposition 4.3. *If $\mathscr{C}\mathscr{C}$-dim $\mathscr{A} < \infty$, then $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$.*

Proof. We claim that for every $X^\bullet \in K^b(\mathscr{A})$ there exists a $\mathscr{C}$-quasi-isomorphism $C_X^\bullet \to X^\bullet$ such that $C_X^\bullet \in K^b(\mathscr{C})$. We proceed by induction on the width $\omega(X^\bullet)$ of $X^\bullet$.

Let $\omega(X^\bullet)$=1. Because $\mathscr{C}$ is contravariantly finite and $\mathscr{C}\mathscr{C}\text{-}\dim \mathscr{A} < \infty$, there exists a $\mathscr{C}$-quasi-isomorphism $C_X^\bullet \to X^\bullet$ with $C_X^\bullet \in K^b(\mathscr{C})$.

Let $\omega(X^\bullet) \geq 2$ with $X^j \neq 0$ and $X^i = 0$ for any $i < j$. Put $X_1^\bullet := X^j[-j-1]$, $X_2^\bullet := \sigma^{>j}X^\bullet$ and $g = d_X^j[-j-1]$. We have a distinguished triangle $X_1^\bullet \xrightarrow{g} X_2^\bullet \to X^\bullet \to X_1^\bullet[1]$ in $K^b(\mathscr{A})$. By the induction hypothesis, there exist $\mathscr{C}$-quasi-isomorphisms f_{X_1}: $C_{X_1}^\bullet \to X_1^\bullet$ and f_{X_2}: $C_{X_2}^\bullet \to X_2^\bullet$ with $C_{X_1}^\bullet$,$C_{X_2}^\bullet \in K^b(\mathscr{C})$. Then by Remark 2.2(1) and Lemma 2.3, f_{X_2} induces an isomorphism:

$$\mathrm{Hom}_{K^b(\mathscr{A})}(C_{X_1}^\bullet, C_{X_2}^\bullet) \cong \mathrm{Hom}_{K^b(\mathscr{A})}(C_{X_1}^\bullet, X_2^\bullet).$$

So there exists a morphism $f^\bullet : C_{X_1}^\bullet \to C_{X_2}^\bullet$, which is unique up to homotopy, such that $f_{X_2} f^\bullet = g f_{X_1}$. Put $C_X^\bullet = \mathrm{Con}(f^\bullet)$. We have the following distinguished triangle in $K^b(\mathscr{C})$:

$$C_{X_1}^\bullet \xrightarrow{f^\bullet} C_{X_2}^\bullet \to C_X^\bullet \to C_{X_1}^\bullet[1].$$

Then there exists a morphism $f_X : C_X^\bullet \to X^\bullet$ such that the following diagram commutes:

$$\begin{tikzcd}
C_{X_1}^\bullet \arrow[r, "f^\bullet"] \arrow[d, "f_{X_1}"] & C_{X_2}^\bullet \arrow[r] \arrow[d, "f_{X_2}"] & C_X^\bullet \arrow[r] \arrow[d, dashed, "f_X"] & C_{X_1}^\bullet[1] \arrow[d, "f_{X_1}[1]"] \\
X_1^\bullet \arrow[r, "g"] & X_2^\bullet \arrow[r] & X^\bullet \arrow[r] & X_1^\bullet[1].
\end{tikzcd}$$

For any $C \in \mathscr{C}$ and any $n \in \mathbb{Z}$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
(C, C_{X_1}^\bullet[n]) \arrow[r] \arrow[d, "{(C,f_{X_1}[n])}"] & (C, C_{X_2}^\bullet[n]) \arrow[r] \arrow[d, "{(C,f_{X_2}[n])}"] & (C, C_X^\bullet[n]) \arrow[r] \arrow[d, dashed, "{(C,f_X[n])}"] & (C, C_{X_1}^\bullet[n+1]) \arrow[r] \arrow[d, "{(C,f_{X_1}[n+1])}"] & (C, C_{X_2}^\bullet[n+1]) \arrow[d, "{(C,f_{X_2}[n+1])}"] \\
(C, X_1^\bullet[n]) \arrow[r] & (C, X_2^\bullet[n]) \arrow[r] & (C, X^\bullet[n]) \arrow[r] & (C, X_1^\bullet[n+1]) \arrow[r] & (C, X_2^\bullet[n+1]),
\end{tikzcd}$$

where $(C, -)$ denotes the functor $\mathrm{Hom}_{K(\mathscr{A})}(C, -)$. Since f_{X_1} and f_{X_2} are $\mathscr{C}$-quasi-isomorphisms, $(C, f_{X_1}[n])$ and $(C, f_{X_2}[n])$ are isomorphisms, and hence so is $(C, f_X[n])$ for each n, that is, f_X is a $\mathscr{C}$-quasi-isomorphism. The claim is proved.

It follows from the claim that every object $X^\bullet$ in $D_{\mathscr{C}}^b(\mathscr{A})$ is isomorphic to some $C_X^\bullet$ of $K^b(\mathscr{C})$ in $D_{\mathscr{C}}^b(\mathscr{A})$. Thus $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$. $\square$

As an application of Proposition 4.3, we have the following

Corollary 4.4. *(1) $\mathscr{C}\mathscr{C}\text{-}\dim M < \infty$ for any $M \in \mathscr{A}$ if and only if $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$.*

(2) If A is of finite representation type, then $\mathscr{C}\mathscr{C}\text{-}\dim \mathscr{A} < \infty$ if and only if $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$.

Proof. In both assertions, the necessity follows from Proposition 4.3. In the following, we only need to prove the sufficiency.

(1) Let $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$ and $M \in \mathscr{A}$. Then $M = 0$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ and M is isomorphic to $C^\bullet$ in $D_{\mathscr{C}}^b(\mathscr{A})$ for some $C^\bullet \in K^b(\mathscr{C})$. We use the equivalent class of right fractions to denote a morphism in $D_{\mathscr{C}}^b(\mathscr{A})$. Let $f/s : C^\bullet \overset{s}{\Longleftarrow} Z^\bullet \xrightarrow{f} M$ be an isomorphism in $D_{\mathscr{C}}^b(\mathscr{A})$, where s is a $\mathscr{C}$-quasi-isomorphism. Then f is a $\mathscr{C}$-quasi-isomorphism. By Lemma 2.4(1), there exists a $\mathscr{C}$-quasi-isomorphism $s^{'} : C^\bullet \to Z^\bullet$. So $fs^{'} : C^\bullet \to M$ is also a $\mathscr{C}$-quasi-isomorphism and hence

$H^i \operatorname{Hom}_{\mathscr{A}}(C, C^\bullet) = 0$ whenever $C \in \mathscr{C}$ and $i \neq 0$. Consider the truncation:

$$C'^{\bullet} := \cdots \to C^{-2} \to C^{-1} \to \operatorname{Ker} d_C^0 \to 0$$

of $C^\bullet$. Then the composition $C'^{\bullet} \hookrightarrow C^\bullet \xrightarrow{fs'} M$ is a $\mathscr{C}$-quasi-isomorphism. Notice that $C^\bullet \in K^b(\mathscr{C})$, we may suppose $C^n \neq 0$ and $C^i = 0$ whenever $i > n$. Then we have a $\mathscr{C}$-acyclic complex $0 \to \operatorname{Ker} d_C^0 \to C^0 \xrightarrow{d_C^0} C^1 \to \cdots \to C^n \to 0$ with all C^i in $\mathscr{C}$. Because $\mathscr{C}$ is closed under direct summands, $\operatorname{Ker} d_C^0 \in \mathscr{C}$ and $\mathscr{C}\mathscr{C}\text{-}\dim M < \infty$.

(2) Let A be of finite representation type, and let $\{M_i \mid 1 \leq i \leq n\}$ be the set of all non-isomorphic indecomposable modules in $\mathscr{A}$. By (1) $\mathscr{C}\mathscr{C}\text{-}\dim M_i < \infty$ for any $1 \leq i \leq n$. Now set $m = \sup\{\mathscr{C}\mathscr{C}\text{-}\dim M_i \mid 1 \leq i \leq n\}$. Since $\mathscr{A}$ is Krull-Schmidt, every module $M \in \mathscr{A}$ can be decomposed into a finite direct sum of modules in $\{M_i \mid 1 \leq i \leq n\}$. Then it is easy to see that $\mathscr{C}\mathscr{C}\text{-}\dim M \leq m$ and $\mathscr{C}\mathscr{C}\text{-}\dim \mathscr{A} \leq m < \infty$. $\square$

As a consequence of Corollary 4.4(1), we have the following

Corollary 4.5. *If A is Gorenstein, then $D_{\mathscr{G}(A)\text{-}sg}(\mathscr{A}) = 0$.*

Proof. Let A be Gorenstein. Because A-proj $\subseteq \mathscr{G}(A)$, we have that $\mathscr{G}(A)$ is admissible in A-mod by [EJ2, Remark 11.5.2]. By [Hos, Theorem], we have $\mathscr{G}(A)\text{-}\dim M < \infty$ for any $M \in \mathscr{A}$. So $D_{\mathscr{G}(A)\text{-}sg}(\mathscr{A}) = 0$ by [AvM, Proposition 4.8] and Corollary 4.4(1). $\square$

Put $\mathscr{G}(\mathscr{C}) = \{M \cong \operatorname{Im}(C^{-1} \to C^0) \mid$ there exists an acyclic complex $\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots$ in $\mathscr{C}$, which is both $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\operatorname{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact$\}$, see [SSW], where it is called the *Gorenstein category* of $\mathscr{C}$. This notion unifies the following ones: modules of Gorenstein dimension zero ([AuB]), Gorenstein projective modules, Gorenstein injective modules ([EJ1]), V-Gorenstein projective modules, V-Gorenstein injective modules ([EJL]), and so on. Set $\mathscr{G}^1(\mathscr{C}) = \mathscr{G}(\mathscr{C})$ and inductively set $\mathscr{G}^n(\mathscr{C}) = \mathscr{G}(\mathscr{G}^{n-1}(\mathscr{C}))$ for any $n \geq 2$. It was shown in [SSW] that $\mathscr{G}(\mathscr{C})$ possesses many nice properties when $\mathscr{C}$ is self-orthogonal. For example, in this case, $\mathscr{G}(\mathscr{C})$ is closed under extensions and $\mathscr{C}$ is a projective generator and an injective cogenerator for $\mathscr{G}(\mathscr{C})$, which induce that $\mathscr{G}^n(\mathscr{C}) = \mathscr{G}(\mathscr{C})$ for any $n \geq 1$, see [SSW] for more details. Later on, Huang generalized this result to an arbitrary full and additive subcategory $\mathscr{C}$ of $\mathscr{A}$, see [Hu].

Denote by ε the class of all $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact complexes of the form: $0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$ with $L, M, N \in \mathscr{G}(\mathscr{C})$. We have the following fact.

Proposition 4.6. *$(\mathscr{G}(\mathscr{C}), \varepsilon)$ is an exact category.*

Proof. We will prove that all the axioms in Definition 2.6 are satisfied. It is trivial that the axiom [E0] is satisfied. In the following, we prove that the other axioms are satisfied.

For [E1^{op}], let $f: G_1 \to G_2$ and $g: G_2 \to G_3$ be admissible epics in $\mathscr{G}(\mathscr{C})$. Then it is easy to see that gf is also an admissible epic. By Lemma 3.7(3), the following $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact sequence:

$$0 \to \operatorname{Ker} gf \to G_1 \xrightarrow{gf} G_3 \to 0$$

is also $\operatorname{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact. It follows from [Hu, Proposition 4.7] that $\operatorname{Ker} gf \in \mathscr{G}(\mathscr{C})$.

For [E2^{op}], let $f : G_2 \to G_3$ be an admissible epic in $\mathscr{G}(\mathscr{C})$ and $g : G_2^{'} \to G_3$ an arbitrary morphism in $\mathscr{G}(\mathscr{C})$. We have the following pull-back diagram with the second row in ε:

$$\begin{tikzcd}
0 \arrow[r] & G_1 \arrow[r, "h^{'}"] \arrow[d, equal] & X \arrow[r, "f^{'}"] \arrow[d, "g^{'}"] & G_2^{'} \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & G_1 \arrow[r, "h"] & G_2 \arrow[r, "f"] & G_3 \arrow[r] & 0.
\end{tikzcd}$$

For any $C \in \mathscr{C}$ and any morphism $\varphi : C \to G_2^{'}$, there exists a morphism $\phi : C \to G_2$ such that $g\varphi = f\phi$. Notice that the right square is a pull-back diagram, so there exists a morphism $\phi^{'} : C \to X$ such that $\varphi = f^{'}\phi^{'}$ and hence the exact sequence $0 \to G_1 \xrightarrow{h^{'}} X \xrightarrow{f^{'}} G_2^{'} \to 0$ is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact. It follows from Lemma 3.7(3) that this sequence is also $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact. By [Hu, Proposition 4.7], $X \in \mathscr{G}(\mathscr{C})$, which implies that $0 \to G_1 \xrightarrow{h^{'}} X \xrightarrow{f^{'}} G_2^{'} \to 0$ lies in ε.

For [E2], let $f : G_1 \to G_2$ be an admissible monic in $\mathscr{G}(\mathscr{C})$ and $g : G_1 \to G_2^{'}$ an arbitrary morphism in $\mathscr{G}(\mathscr{C})$. We have the following push-out diagram with the first row in ε:

$$\begin{tikzcd}
0 \arrow[r] & G_1 \arrow[r, "f"] \arrow[d, "g"] & G_2 \arrow[r, "h"] \arrow[d, "g^{'}"] & G_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_2^{'} \arrow[r, "f^{'}"] & D \arrow[r, "h^{'}"] & G_3 \arrow[r] & 0.
\end{tikzcd}$$

For any $C \in \mathscr{C}$ and any morphism $\varphi : C \to G_3$, there exists a morphism $\phi : C \to G_2$ such that $\varphi = h\phi = h^{'}g^{'}\phi$. So the exact sequence $0 \to G_2^{'} \xrightarrow{f^{'}} D \xrightarrow{h^{'}} G_3 \to 0$ is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact. It follows from Lemma 3.7(3) that this sequence is also $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact. By [Hu, Proposition 4.7], $D \in \mathscr{G}(\mathscr{C})$, which implies that $0 \to G_2^{'} \xrightarrow{f^{'}} D \xrightarrow{h^{'}} G_3 \to 0$ lies in ε .

Now let $0 \to G_0 \xrightarrow{i} G_1 \to G_2 \to 0$ and $0 \to G_1 \xrightarrow{j} G_1^{'} \to G_1^{''} \to 0$ lie in ε. We have the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G_0 \arrow[r, "i"] \arrow[d, equal] & G_1 \arrow[r] \arrow[d, "j"] & G_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_0 \arrow[r, "ji"] & G_1^{'} \arrow[r] \arrow[d] & G_2^{'} \arrow[r] \arrow[d] & 0 \\
 & & G_1^{''} \arrow[r, equal] \arrow[d] & G_1^{''} \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

By [E2], the rightmost column lies in ε. For any $C \in \mathscr{C}$, applying the functor $(C, -) := \mathrm{Hom}_{\mathscr{A}}(C, -)$

to the commutative diagram we get the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (C,G_0) \arrow[r, "(C,i)"] \arrow[d, equal] & (C,G_1) \arrow[r] \arrow[d, "(C,j)"] & (C,G_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (C,G_0) \arrow[r, "(C,ji)"] & (C,G_1^{'}) \arrow[r] \arrow[d] & (C,G_2^{'}) \arrow[d] & \\
 & & (C,G_1^{''}) \arrow[r, equal] \arrow[d] & (C,G_1^{''}) \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

By the snake lemma, the morphism $(C,G_1^{'}) \to (C,G_2^{'})$ is epic. Then $0 \to G_0 \xrightarrow{ji} G_1^{'} \to G_2^{'} \to 0$ lies in ε, and [E1] follows. $\square$

By Proposition 4.6, we have the following

Corollary 4.7. $(\mathscr{G}(\mathscr{C}), \varepsilon)$ is a Frobenius category, that is, $(\mathscr{G}(\mathscr{C}), \varepsilon)$ has enough projective objects and enough injective objects such that the projective objects coincide with the injective objects.

Proof. Because $\mathscr{C}$ is the class of (relative) projective-injective objects in $\mathscr{G}(\mathscr{C})$, the assertion follows from Proposition 4.6. $\square$

For $M, N \in \mathscr{A}$, let $\mathscr{C}(M,N)$ denote the subspace of A-maps from M to N factoring through $\mathscr{C}$. Put ${}^{\perp_{\mathscr{C}}}\mathscr{C} = \{M \in \mathscr{A} \mid \mathrm{Ext}^i_{\mathscr{C}}(M,C) = 0 \text{ for any } C \in \mathscr{C} \text{ and } i \geq 1\}$. By definition, it is clear that $\mathscr{C} \subseteq \mathscr{G}(\mathscr{C}) \subseteq {}^{\perp_{\mathscr{C}}}\mathscr{C}$.

Lemma 4.8. *For any $M \in {}^{\perp_{\mathscr{C}}}\mathscr{C}$ and $N \in \mathscr{A}$, we have a canonical isomorphism of abelian groups:*

$$\mathrm{Hom}_{\mathscr{A}}(M,N)/\mathscr{C}(M,N) \cong \mathrm{Hom}_{D_{\mathscr{C}\text{-}sg}(\mathscr{A})}(M,N).$$

Proof. In the following, a morphism from M to N in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ is denoted by the equivalent class of left fractions $s\backslash a : M \xrightarrow{a} Z^\bullet \overset{s}{\Longleftarrow} N$, where $Z^\bullet \in D^b_{\mathscr{C}}(\mathscr{A})$ and $\mathrm{Con}(s) \in K^b(\mathscr{C})$. We have a distinguished triangle in $D^b_{\mathscr{C}}(\mathscr{A})$:

$$N \overset{s}{\Longrightarrow} Z^\bullet \to \mathrm{Con}(s) \to N[1]. \tag{1}$$

Consider the canonical map $G : \mathrm{Hom}_{\mathscr{A}}(M,N) \to \mathrm{Hom}_{D_{\mathscr{C}\text{-}sg}(\mathscr{A})}(M,N)$ defined by $G(f) = \mathrm{id}_N \backslash f$. We first prove that G is surjective. For any $N \in \mathscr{A}$, we have the following left $\mathscr{C}$-resolution of N:

$$\cdots \to C^{-n} \xrightarrow{d_C^{-n}} C^{-n+1} \to \cdots \xrightarrow{d_C^{-1}} C^0 \xrightarrow{d_C^0} N \to 0.$$

Then in $D_{\mathscr{C}}(\mathscr{A})$, N is isomorphic to the complex $C^\bullet := \cdots \to C^{-n} \xrightarrow{d_C^{-n}} C^{-n+1} \to \cdots \xrightarrow{d_C^{-1}} C^0 \to 0$, and so is isomorphic to the complex $0 \to \operatorname{Ker} d_C^{-l} \to C^{-l} \xrightarrow{d_C^{-l}} C^{-l+1} \to \cdots \xrightarrow{d_C^{-1}} C^0 \to 0$ for any $l \geq 0$. Hence we have a distinguished triangle in $D^b_{\mathscr{C}}(\mathscr{A})$:

$$\operatorname{Ker} d_C^{-l}[l] \to \sigma^{\geq -l} C^\bullet \xrightarrow{d_C^0} N \overset{s'}{\Longrightarrow} \operatorname{Ker} d_C^{-l}[l+1], \tag{2}$$

where $\operatorname{Con}(s') \in K^b(\mathscr{C})$. Since $\operatorname{Con}(s) \in K^b(\mathscr{C})$, it follows from Proposition 3.3 that there exists $l_0 \gg 0$ such that for any $l \geq l_0$, we have

$$\operatorname{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(\operatorname{Con}(s), \operatorname{Ker} d_C^{-l}[l+1]) = 0.$$

Take $l = l_0$ in (2). On one hand, applying the functor $\operatorname{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(-, \operatorname{Ker} d_C^{-l_0}[l_0+1])$ to (1) we get $h : Z^\bullet \to \operatorname{Ker} d_C^{-l_0}[l_0+1]$ such that $s' = hs$. So we have $s \backslash a = s' \backslash (ha)$. On the other hand, applying $\operatorname{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(M, -) := (M, -)$ to (2) we get an exact sequence

$$(M, N) \xrightarrow{(M,s')} (M, \operatorname{Ker} d_C^{-l_0}[l_0+1]) \to (M, (\sigma^{\geq -l_0} C^\bullet)[1]).$$

Since $M \in {}^{\perp_{\mathscr{C}}}\mathscr{C}$, by using induction on $\omega(\sigma^{\geq -l_0} C^\bullet)$ we have $(M, (\sigma^{\geq -l_0} C^\bullet)[1])$=0, and hence there exists $f : M \to N$ such that ha=$s'f$. Therefore we have $s\backslash a$=$s'\backslash(ha)$=$s'\backslash(s'f)$=$\mathrm{id}_N \backslash f$, that is, G is surjective.

Next, if $f : M \to N$ satisfies $G(f) = \mathrm{id}_N \backslash f = 0$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$, then there exists $s : N \to Z^\bullet$ with $\operatorname{Con}(s) \in K^b(\mathscr{C})$ such that $sf = 0$ in $D^b_{\mathscr{C}}(\mathscr{A})$. Use the same notations as in (1) and (2), by the above argument we have $s' = hs$, so $s'f = 0$. Applying $\operatorname{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(M, -)$ to (2) we get that there exists $f' : M \to \sigma^{\geq -l_0} C^\bullet$ such that $f = d_C^0 f'$.

Put $\sigma^{<0}(\sigma^{\geq -l_0})C^\bullet := 0 \to C^{-l_0} \to C^{-l_0+1} \to \cdots \to C^{-1} \to 0$. We have the following distinguished triangle:

$$(\sigma^{<0}(\sigma^{\geq -l_0})C^\bullet)[-1] \longrightarrow C^0 \xrightarrow{\pi} \sigma^{\geq -l_0} C^\bullet \to \sigma^{<0}(\sigma^{\geq -l_0})C^\bullet$$

in $D^b_{\mathscr{C}}(\mathscr{A})$, where π is the canonical map. By applying the functor $\operatorname{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(M, -)$ to this triangle, it follows from $M \in {}^{\perp_{\mathscr{C}}}\mathscr{C}$ that $\operatorname{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(M, \sigma^{<0}(\sigma^{\geq -l_0})C^\bullet) = 0$, and hence there exists $g : M \to C^0$ such that $f' = \pi g$. So $f = d_C^0 \pi g$ in $D^b_{\mathscr{C}}(\mathscr{A})$. By Proposition 3.3(3), $\mathscr{A}$ is a full subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$. So f factors through C^0 in $\mathscr{A}$, and hence $\operatorname{Ker} G \subseteq \mathscr{C}(M, N)$. Since $\mathscr{C}(M, N) \subseteq \operatorname{Ker} G$ trivially, $\operatorname{Ker} G = \mathscr{C}(M, N)$, which means that $\operatorname{Hom}_{\mathscr{A}}(M, N)/\mathscr{C}(M, N) \cong \operatorname{Hom}_{D_{\mathscr{C}\text{-}sg}(\mathscr{A})}(M, N)$. □

Let $\theta : \mathscr{G}(\mathscr{C}) \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ be the composition of the following three functors: the embedding functors $\mathscr{G}(\mathscr{C}) \hookrightarrow \mathscr{A}$, $\mathscr{A} \hookrightarrow D^b_{\mathscr{C}}(\mathscr{A})$ and the localization functor $D^b_{\mathscr{C}}(\mathscr{A}) \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$, and let $\underline{\mathscr{G}(\mathscr{C})}$ denote the stable category of $\mathscr{G}(\mathscr{C})$.

Proposition 4.9. *θ induces a fully faithful functor $\theta' : \underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$.*

Proof. Since $\mathscr{G}(\mathscr{C}) \subseteq {}^{\perp_{\mathscr{C}}}\mathscr{C}$, the assertion follows from Lemma 4.8. □

Recall from [C2] that a *∂-functor* is an additive functor F from an exact category $(\mathscr{B}, \varepsilon)$ to a triangulated category $\mathcal{C}$ satisfying that for any short exact sequence $L \xrightarrow{i} M \xrightarrow{p} N$ in ε, there exists

a morphism $\omega_{(i,p)} : F(N) \to F(L)[1]$ such that the the triangle

$$F(L) \xrightarrow{F(i)} F(M) \xrightarrow{F(p)} F(N) \xrightarrow{\omega_{(i,p)}} F(L)[1]$$

in $\mathcal{C}$ is distinguished; moreover, the morphism $\omega_{(i,p)}$ are "functorial" in the sense that any morphism between two short exact sequences in ε:

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[d, "f"] & M \arrow[r, "p"] \arrow[d, "g"] & N \arrow[d, "h"] \\ L' \arrow[r, "i'"] & M' \arrow[r, "p'"] & N', \end{tikzcd}$$

the following is a morphism of triangles:

$$\begin{tikzcd} F(L) \arrow[r, "F(i)"] \arrow[d, "F(f)"] & F(M) \arrow[r, "F(p)"] \arrow[d, "F(g)"] & F(N) \arrow[r, "\omega_{(i,p)}"] \arrow[d, "F(h)"] & F(L)[1] \arrow[d, "F(f)[1]"] \\ F(L') \arrow[r, "F(i')"] & F(M') \arrow[r, "F(p')"] & F(N') \arrow[r, "\omega_{(i',p')}"] & F(L')[1]. \end{tikzcd}$$

By [H1, Chapter I, Theorem 2.6] and Corollary 4.7, $\underline{\mathscr{G}(\mathscr{C})}$ and $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ are triangulated categories. Moreover, we have

Proposition 4.10. *The functor θ' in Proposition 4.9 is a triangle functor.*

Proof. We first claim that θ is a ∂-functor. In fact, let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact complex with all terms in $\mathscr{G}(\mathscr{C})$. Then it induces a distinguished triangle in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$, saying $\theta(L) \xrightarrow{\theta(f)} \theta(M) \xrightarrow{\theta(g)} \theta(N) \xrightarrow{\omega_{(f,g)}} \theta(L)[1]$. It is clear that $\omega_{(f,g)}$ is "functorial" . This shows that θ is a ∂-functor.

Note that every object in $\mathscr{C}$ is zero in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. So θ vanishes on the projective-injective objects in $\mathscr{G}(\mathscr{C})$. It follows from [C2, Lemma 2.5] that the induced functor θ' is a triangle functor. $\square$

By Propositions 4.9 and 4.10 the natural triangle functor $\underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ is fully faithful. It is of interest to make sense when it is essentially surjective (or dense). We have the following

Theorem 4.11. *If $\mathscr{C}\mathscr{G}(\mathscr{C})$-dim $\mathscr{A} < \infty$, then the natural functor $\theta : \mathscr{G}(\mathscr{C}) \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ is essentially surjective (or dense).*

Proof. Let $X^\bullet \in D^b_{\mathscr{C}}(\mathscr{A})$. By Proposition 3.4, there exists $C_0^\bullet = (C_0^i, d^i_{C_0}) \in K^{-,\mathscr{C}b}(\mathscr{C})$ such that $X^\bullet \cong C_0^\bullet$ in $D^b_{\mathscr{C}}(\mathscr{A})$. So there exists $n_0 \in \mathbb{Z}$ such that $H^i(\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, C_0^\bullet)) = 0$ for any $i \leq n_0$. Let $K^i = \mathrm{Ker}\, d^i_{C_0}$. Then $C_0^\bullet$ is isomorphic to the complex:

$$0 \to K^i \to C_0^i \xrightarrow{d^i_{C_0}} C_0^{i+1} \xrightarrow{d^{i+1}_{C_0}} C_0^{i+2} \to \cdots$$

in $D^b_{\mathscr{C}}(\mathscr{A})$ for any $i \leq n_0$. It induces a distinguished triangle in $D^b_{\mathscr{C}}(\mathscr{A})$, hence a distinguished triangle in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ of the following form:

$$K^i[-i] \to \sigma^{\geq i} C_0^\bullet \to C_0^\bullet \to K^i[-i+1].$$

Since $\sigma^{\geq i}C_0^\bullet \in K^b(\mathscr{C})$, $C_0^\bullet \cong K^i[-i+1]$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. Take $l_0 = i$ and $Y = K^i$. Then $C_0^\bullet \cong Y[-l_0+1]$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. By assumption we may assume that $\mathscr{C}\mathscr{G}(\mathscr{C})\text{-}\dim Y = m_0 < \infty$. Let $C_1^\bullet \to Y$ be the left $\mathscr{C}$-resolution of Y. We claim that for any $n \leq -m_0+1$, $\operatorname{Ker} d_{C_1}^n \in \mathscr{G}(\mathscr{C})$, where $d_{C_1}^n$ is the nth differential of $C_1^\bullet$.

We have a $\mathscr{C}$-acyclic complex:

$$0 \to G^{-m_0} \to G^{-m_0+1} \to \cdots \to G^{-1} \to G^0 \to Y \to 0$$

with $G^j \in \mathscr{G}(\mathscr{C})$ for any $-m_0 \leq j \leq 0$. Let $G^\bullet$ be the complex $0 \to G^{-m_0} \to G^{-m_0+1} \to \cdots \to G^{-1} \to G^0 \to 0$. By Lemma 2.3, there exists a $\mathscr{C}$-quasi-isomorphism $C_1^\bullet \to G^\bullet$ lying over id_Y, and hence its mapping cone is $\mathscr{C}$-acyclic. So for any $n \leq -m_0+1$, we get the following $\mathscr{C}$-acyclic complex:

$$0 \to \operatorname{Ker} d_{C_1}^n \to C_1^n \to \cdots \to C_1^{-m_0} \to C_1^{-m_0+1} \oplus G^{-m_0} \to \cdots \to C_1^0 \oplus G^{-1} \to G^0 \to 0.$$

Note that this complex is acyclic because $\mathscr{C}$ is admissible. Put $K = \operatorname{Ker}(C_1^0 \oplus G^{-1} \to G^0)$, we get a $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence $0 \to K \to C_1^0 \oplus G^{-1} \to G^0 \to 0$. By Lemma 3.7(3), we get an exact sequence:

$$0 \to \operatorname{Hom}_{\mathscr{A}}(G^0, C) \to \operatorname{Hom}_{\mathscr{A}}(C_1^0 \oplus G^{-1}, C) \to \operatorname{Hom}_{\mathscr{A}}(K, C) \to \operatorname{Ext}^1_{\mathscr{C}}(G^0, C)$$

for any $C \in \mathscr{C}$. Since $G^0 \in \mathscr{G}(\mathscr{C})$, $\operatorname{Ext}^1_{\mathscr{C}}(G^0, C) = 0$ and so the exact sequence $0 \to K \to C_1^0 \oplus G^{-1} \to G^0 \to 0$ is $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{C})$-exact. Because both $C_1^0 \oplus G^{-1}$ and G^0 are in $\mathscr{G}(\mathscr{C})$, $K \in \mathscr{G}(\mathscr{C})$ by [Hu, Proposition 4.7]. Iterating this process, we get that $\operatorname{Ker} d_{C_1}^n \in \mathscr{G}(\mathscr{C})$ for any $n \leq -m_0+1$. The claim is proved.

Choose a left $\mathscr{C}$-resolution $C_1^\bullet$ of Y and put $X = \operatorname{Ker} d_{C_1}^{-m_0+1}$. By the above claim we have a $\mathscr{C}$-acyclic complex:

$$0 \to X \to C_1^{-m_0+1} \to C_1^{-m_0+2} \to \cdots \to C_1^0 \to Y \to 0$$

with $X \in \mathscr{G}(\mathscr{C})$. Then $Y \cong X[m_0]$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ and $X^\bullet \cong C_0^\bullet \cong Y[-l_0+1] \cong X[m_0 - l_0 + 1]$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. We may assume that $X^\bullet \cong C_0^\bullet \cong X[r_0]$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ for $r_0 > 0$. Because $X \in \mathscr{G}(\mathscr{C})$, we get a $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence $0 \to X \to C^0 \to C^1 \to \cdots \to C^{r_0-1} \to X' \to 0$ with $X' \in \mathscr{G}(\mathscr{C})$ and $C^i \in \mathscr{C}$ for any $0 \leq i \leq r_0 - 1$. It follows that $X \cong X'[-r_0]$ and $X^\bullet \cong C_0^\bullet \cong X[r_0] \cong X'$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. This completes the proof. $\square$

The following is the main result of this paper.

Theorem 4.12. *If $\mathscr{C}\mathscr{G}(\mathscr{C})\text{-}\dim \mathscr{A} < \infty$, then the natural functor $\theta : \mathscr{G}(\mathscr{C}) \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ induces a triangle-equivalence $\theta' : \underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$.*

Proof. It follows directly from Propositions 4.9, 4.10 and Theorem 4.11. $\square$

The following result is the dual version of Happel's result, see [H2, Theorem 4.6].

Corollary 4.13. *If A is Gorenstein, then the canonical functor $\mathscr{G}(A) \to D_{sg}(A)$ induces a triangle-equivalence $\underline{\mathscr{G}(A)} \to D_{sg}(A)$.*

Proof. Let A be Gorenstein and $\mathscr{C} = A$- proj. Then $\mathscr{C}\mathscr{G}(\mathscr{C})$-dim $\mathscr{A} < \infty$ by [Hos, Theorem]. Now the assertion is an immediate consequence of Theorem 4.12. $\square$

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